

Complexation of Wavenumbers in Solitons

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Abstract

It is shown that, by letting wavenumbers and frequencies complex in Hirota's bilinear method, new classes of exact solutions of soliton equations can be obtained systematically. They include not only singular or N -homoclinic solutions but also N -wavepacket solutions.

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Recent awareness of homoclinic solutions of soliton equations [1,2] has made it possible to apply them for studies of chaotic phenomena in nonlinear, infinite dynamical and complex systems, e.g., [3]. In contrast to the low-dimensional ordinary differential equations, partial differential equations may have multiple or infinite homoclinic structures. Therefore, it is important to find and classify homoclinic solutions systematically.

The key idea of this paper is to make wavenumbers complex in soliton solutions. Usually this procedure fails because the dependent variable becomes complex for real-valued equations. However, if we keep the complex conjugate pair for wavenumbers, the difficulty can be avoided.

Let us start with the simplest system, the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0, \quad (1)$$

where $u_x = \partial_x u = \partial u / \partial x$. Letting $u(x, t) = 2(\ln f(x, t))_{xx}$, the bilinear form becomes [4]

$$Mf \cdot f = 0, \quad (2)$$

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where

$$M = M(D_t, D_x) = D_x(D_t + D_x^3), \quad (3)$$

and D_x is the Hirota's bilinear operator,

$$D_x f \cdot g = (\partial_x - \partial_{x'}) f(x) g(x')|_{x'=x}. \quad (4)$$

The N -soliton solution is given by

$$f = \sum_{\mu=0,1} \exp \left(\sum_{j=1}^N \mu_j \eta_j + \sum_{j>l}^N \mu_j \mu_l A_{jl} \right), \quad (5)$$

with

$$\eta_j = k_j x + \Omega_j t + \eta_{0j}, \quad (6)$$

$$\Omega_j = -k_j^3, \quad (7)$$

$$\exp A_{jl} \equiv a_{jl} = -\frac{M(\Omega_j - \Omega_l, k_j - k_l)}{M(\Omega_j + \Omega_l, k_j + k_l)}, \quad (8)$$

where $\sum_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_i = 0, 1$ for $i = 1, \dots, N$.

This solution gives solitons for real k_j . If we make k_j complex conjugate as $k_{j-1} = k_j^*$ for $j = 2, 4, \dots, 2M$ and real for $j = 2M + 1, \dots, N$, it no longer gives classical solitons. We note that the proof of the N -soliton solutions by mathematical induction [4] is still valid for complex wavenumbers.

For $M = 1$ and $L \equiv N - 2M = 0$, the function f and the solution can be described by

$$f = 1 + 2 \exp \xi_1 \cos \zeta_1 + a_{12} \exp(2\xi_1), \quad (9)$$

$$u = 4K_1^2 \frac{e^{-\xi_1} \cos(\zeta_1 + 2\varphi_1) + a_{12} e^{\xi_1} \cos(\zeta_1 - 2\varphi_1) + 2[a_{12} \cos^2 \varphi_1 - \sin^2 \varphi_1]}{(e^{-\xi_1} + a_{12} e^{\xi_1} + 2 \cos \zeta_1)^2}, \quad (10)$$

where ξ_i and ζ_i are the real and imaginary parts of η_i , and

$$a_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 = -\tan^2 \varphi_1, \quad (11)$$

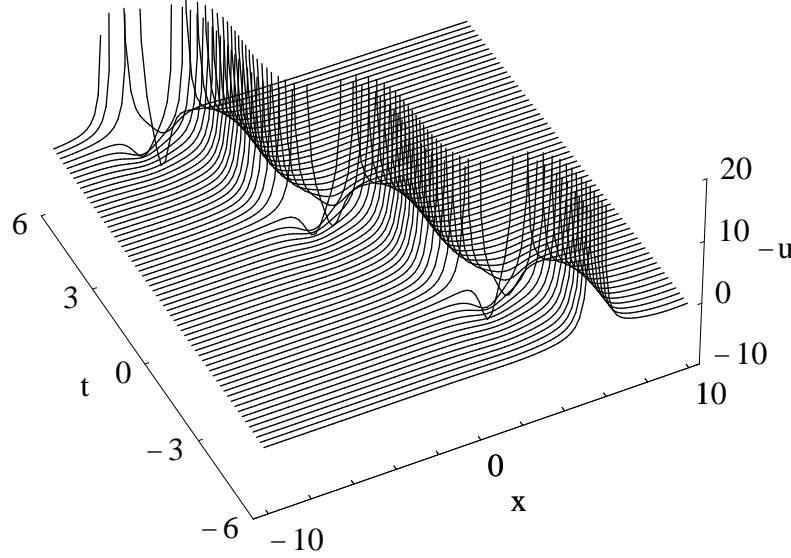


Fig. 1. The 3D plot of a one-singular-point solution of the KdV equation with $k_1 = \exp(i\pi/4)$. The constants η_{i0} are zero throughout the illustrative examples in this paper.

for $k_1 = K_1 \exp i\varphi_1$. Since a_{12} is negative, it can be shown that this solution possesses one singular point ($f = 0$) in x for given t . Figure 1 shows the 3D plot of this solution.

It looks as the singular point travels at the velocity $-\Omega_{1r}/k_{1r}$ (subscript r denotes the real part) with oscillating back and forth in the stripe surrounded by lines $\xi_1 = [\ln(\sqrt{1-a_{12}}) \pm 1]/(-a_{12})$, which are derived by $f = 0$ and $\cos \zeta_1 = \pm 1$ in (9). The singularity behaves as $u \sim u_s = -2(x - x_0)^{-2}$, which is derived by a simple zero of f in x . It should be noted that u_s is an exact steady solution of the KdV equation. It is also observed that a small pulse with the sign opposite to the singularity is travelling with the velocity $-\Omega_{1i}/k_{1i}$.

The interaction between two singular points for $(M, L) = (2, 0)$ is illustrated in Figure 2. The asymptotic behavior of the singular point for k_1 can be given as follows. In the moving frame $X_1 = x + (\Omega_{1r}/k_{1r})t$, η_3 can be written as

$$\eta_3 = k_{3r}X_1 + \sigma_{3r}t + i(k_{3i}X_1 + \sigma_{3i}t) + \eta_{30}, \quad (12)$$

where $\sigma_{3r} = \Omega_{3r} - k_{3r}\Omega_{1r}/k_{1r}$ and $\sigma_{3i} = \Omega_{3i} - k_{3i}\Omega_{1r}/k_{1r}$. If $\sigma_{3r} < 0$,

$$f \sim 1 + e^{\eta_1} + e^{\eta_2} + a_{12}e^{\eta_1+\eta_2} = 0, \quad (13)$$

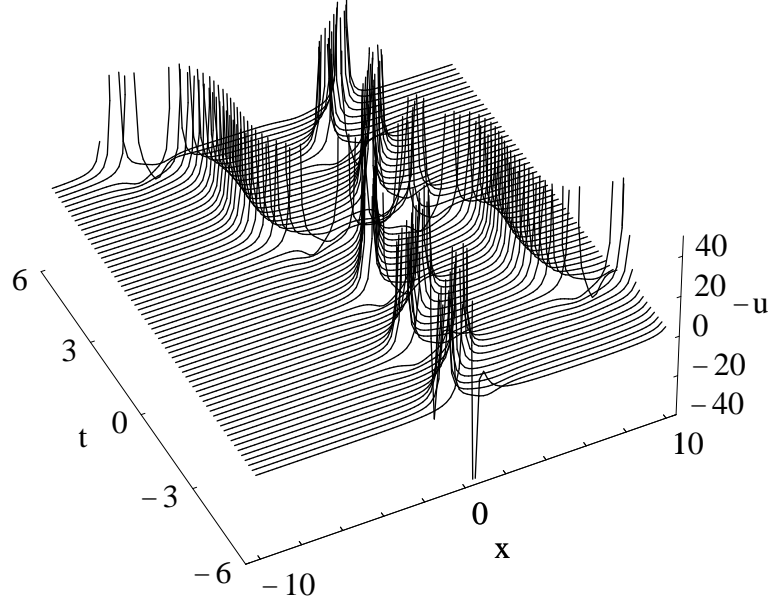


Fig. 2. The 3D plot of the interaction between two singular points of the KdV equation with $k_1 = \exp(i\pi/4)$ and $k_2 = 2^{1/2} \exp(i\pi/6)$.

as $t \rightarrow \infty$. Similarly, in the moving frame $X_3 = x + (\Omega_{3r}/k_{3r})t$,

$$\eta_1 = k_{1r}X_3 + \sigma_{1r}t + i(k_{1i}X_3 + \sigma_{1i}t) + \eta_{10}, \quad (14)$$

where $\sigma_{1r} = -k_{1r}\sigma_{3r}/k_{3r}$ and $\sigma_{1i} = \Omega_{1i} - k_{1i}\Omega_{3r}/k_{3r}$. Assuming $k_{1r}/k_{3r} > 0$, we have $\sigma_{1r} > 0$ and

$$f \sim a_{12}e^{\eta_1+\eta_2}[1 + e^{\eta_3+\delta_3} + e^{\eta_4+\delta_4} + a_{34}e^{\eta_3+\eta_4+\delta_3+\delta_4}], \quad (15)$$

as $t \rightarrow \infty$, where the *complex* phase shift is given by $\delta_3 = \delta_4^* = \ln(a_{13}a_{23})$.

In the limit $t \rightarrow -\infty$ and with the same assumption as the above, the moving frames X_1 and X_3 lead to the asymptotic forms (15) and (13) with replacement $(1, 2) \leftrightarrow (3, 4)$ and the phase shift $\delta_1 = \delta_2^* = \ln(a_{13}a_{14})$. From the definition of a_{ij} , we have $\delta_1 = \delta_3$ and the conservation law of the total phase shift holds, i.e., $\delta_1 + \delta_2 = \delta_3 + \delta_4$. The conservation law does not depend on the signs of k_{1r}/k_{3r} and σ_{1r} .

Figure 3 shows the interaction between a singular point and a soliton for $(M, L) = (1, 1)$. Under the same assumptions of the signs of k_{1r}/k_{3r} and σ_{1r} ,

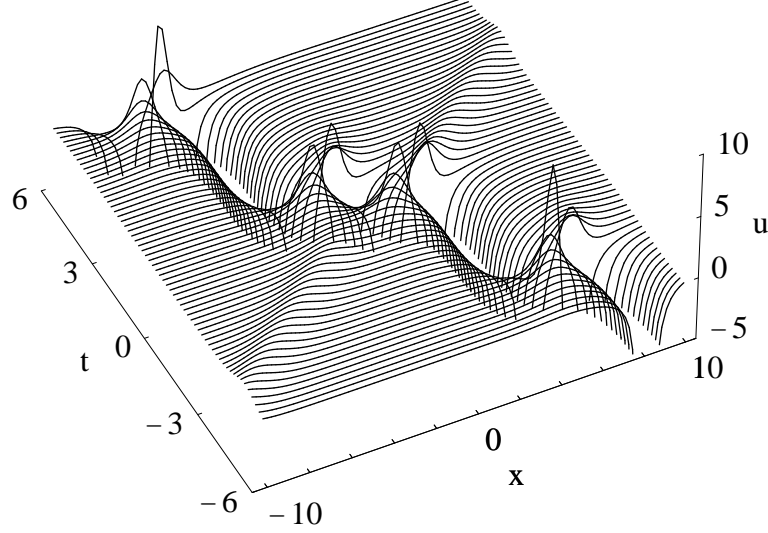


Fig. 3. The 3D plot of the interaction between a singular point and a soliton of the KdV equation with $k_1 = \exp(i\pi/4)$ and $k_2 = 2^{1/2}$.

the asymptotic behavior of the soliton is

$$u \sim \frac{k_{3r}^2}{2} \operatorname{sech}^2 \frac{\xi_3}{2}, \quad (16)$$

as $t \rightarrow -\infty$ and

$$u \sim \frac{k_{3r}^2}{2} \operatorname{sech}^2 \left(\frac{\xi_3}{2} + \delta_3 \right), \quad (17)$$

as $t \rightarrow \infty$, where the phase shift is given by $\delta_3 = \ln(a_{13}a_{23})$. Similarly, the behavior of the singular point is determined by the zero of (13) and (15) with $(1, 2) \leftrightarrow (3, 4)$, the factor in front of the square bracket in (15) replaced by e^{η_3} , and the phase shift $\delta_1 = \delta_2^* = \ln a_{13}$ as $t \rightarrow \infty$ and $t \rightarrow -\infty$, respectively. Again, the total phase shift is conserved.

As we see in the above examples, the singular points behave as if they keep their identities after they collide with each other like solitons. Therefore, they may be called *singulons*.

At first sight, it seems that this singularity makes the solutions physically meaningless. However, the singularity shown above does not always appear.

As such an example, the Boussinesq equation of the form

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{4x} = 0, \quad (18)$$

is considered. Substituting $u = 2(\ln f)_{xx}$, the bilinear equation [5] is given by (2) with the operator

$$M = D_t^2 - D_x^2 - D_x^4. \quad (19)$$

The solution is given by (5) with the dispersion relation

$$\Omega_i = \pm k_i \sqrt{1 + k_i^2}. \quad (20)$$

We note that this solution with complex wavenumbers corresponds to a certain limit of the periodic solutions by two-dimensional θ -function (3.1) of [6] with $\eta_1 = \eta_2^*$, although it is implicit that the wavenumber and the frequency may be complex in [6].

If we choose pure imaginary wavenumbers $k_{j-1} = -k_j = iK_{j-1}$ and $K_j > 1$ for $j = 2, \dots, 2M$ and $L = 0$, we obtain so-called homoclinic solutions, analogous to homoclinic orbits in ordinary differential equations. Strictly speaking, they are temporally homoclinic and spatially periodic (or quasi-periodic) solutions.

As an example, let us consider the simplest complexation, i.e. $(M, L) = (1, 0)$. The asymptotic behaviors of the solution given by (10) are

$$u \sim 4K_1^2 \exp \xi_1 \cos(\zeta_1 + 2\varphi_1), \quad (21)$$

as $t \rightarrow -\infty$ and

$$u \sim 4K_1^2/a_{12} \exp(-\xi_1) \cos(\zeta_1 - 2\varphi_1), \quad (22)$$

as $t \rightarrow \infty$, assuming that Ω_{1r} is positive.

If we put $k_1 = -k_2 = iK_1$ and $K_1 > 1$, the frequency Ω_j becomes real and can be chosen as $\Omega_1 = \Omega_2 = K_1 \sqrt{K_1^2 - 1}$. In this case, the interaction coefficient is $a_{12} = (4K_1^2 - 1)/(K_1^2 - 1)$ and positive. Since $a_{12} > 4$ and f can be written as

$$f = 2\sqrt{a_{12}} \cosh(\xi_1 + \delta_1) + 2 \cos \zeta_1, \quad (23)$$

with $\delta_1 = (\ln a_{12})/2$, this solution remains finite for all x and t .

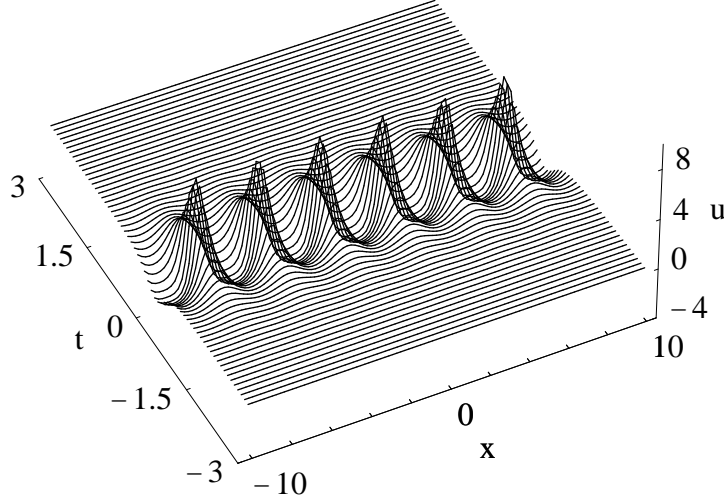


Fig. 4. The 3D plot of the one-homoclinic solution of the Boussinesq equation with $k_1 = 2i$.

As $t \rightarrow -\infty$, the solution tends to a homoclinic point $u = 0$ and behaves as $u \sim -4K_1^2 \exp \xi_1 \cos \zeta_1$, where $\xi_1 = \Omega_1 t + \Re(\eta_{10})$ and $\zeta_1 = K_1 x + \Im(\eta_{10})$. Similarly, as $t \rightarrow \infty$, $u \sim -4K_1^2/a_{12} \exp(-\xi_1) \cos \zeta_1$. This behavior is no more than the linear approximation of the Boussinesq equation around $u = 0$. Here a_{12} plays a role of shift of time and the offset of the amplitude. Figure 4 shows the 3D plot of the one-homoclinic solution.

One remarkable feature of the Boussinesq equation with the spatial period l_x comparing with the nonlinear Schrödinger equation [1] is that there is a countably infinite number of homoclinic orbits due to the short wavelength instability $k_j = iK_j$ with $K_j = 2\pi j/l_x > 1$.

The necessary and sufficient condition for the regularity of the $(M, L) = (1, 0)$ solution is $a_{12} > 1$. Using

$$a_{12} = \frac{|1 + K_1^2 e^{2i\varphi}| - 1 - 4K_1^2 \cos 2\varphi_1 + 3K_1^2}{|1 + K_1^2 e^{2i\varphi}| - 1 - 4K_1^2 \cos 2\varphi_1 - 3K_1^2}, \quad (24)$$

and $K_1 > 1$, the condition can be summarized as

$$\cos 2\varphi_1 < -\frac{3}{4} - \frac{1}{4K_1^2}. \quad (25)$$

For a complex wavenumber k_1 , this solution may no longer be called homo-

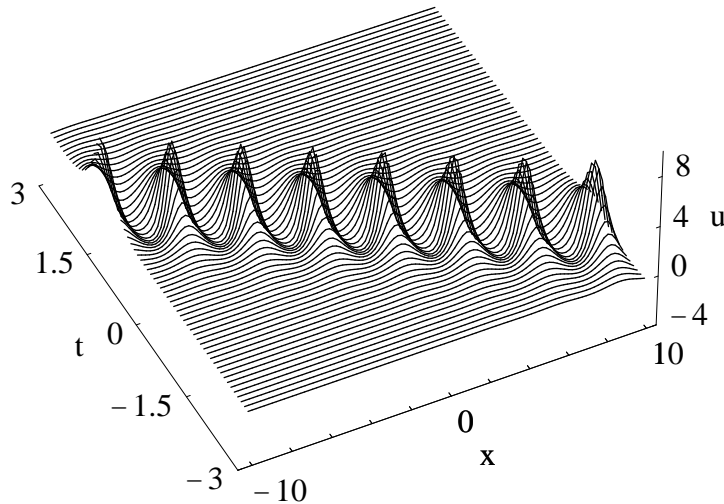


Fig. 5. The 3D plot of the one-wavepacket solution of the Boussinesq equation with a complex wavenumber $k_1 = 2 \exp(i\varphi_1)$, $\varphi_1 = \cos^{-1}(1/4)$.

clinic. Figure 5 shows the 3D plot of the solution with a complex wavenumber k_1 satisfying the condition (25).

The solution shows that a wavepacket travelling at the group velocity $-\Omega_{1r}/k_{1r}$ of waves with the phase velocity $-\Omega_{1r}/k_{1i}$. Therefore, it may be called a *one-wavepacket* solution. Similarly, we may construct two- and N -wavepacket solutions, and interactions between solitons, homoclinic solutions and wavepackets.

We have shown that the Boussinesq equation has richer geometrical structures in phase space than the KdV equation, although the homoclinic solutions may be far from modelling shallow water waves because of the assumption $u_t \approx -u_x$ in the derivation of the equations [7]. The complexation of wavenumbers in solitons shown in this paper has a wide application and is expected to give a tool for finding a new class of exact solutions.

References

- [1] B. M. Herbst and M. J. Ablowitz, Phys. Rev. Lett. 62 (1989) 2065.
- [2] M. Umeki, RIMS kokyuroku 974 (1996)
- [3] A. R. Bishop, M. G. Forest, D. W. McLaughlin and E. A. Oberman, Phys. Lett A 126 (1988) 335.

- [4] R. Hirota, Phys. Rev. Lett. 27 (1971) 1192.
- [5] R. Hirota, J. Math. Phys. 14 (1973) 810.
- [6] A. Nakamura, J. Phys. Soc. Jpn. 47 (1979) 1701.
- [7] J. W. Miles Ann. Rev. Fluid Mech. 12 (1980) 11.